Let us define the bit allocation problem for one dimensional signal (from here on will be called signal) we will be discussing in this report.

Let  be a signal where . We want to represent the signal using a bit budget of  bits. To do so, we will first sample the signal, and then quantize the samples.

**Error Measurement**

We would like to quantify the error of our approximation, so we can measure the quality of an approximation. We will use the MSE (mean squared error) to measure how good an approximation is. Given a signal  and an approximation  of the signal, the MSE of the approximation over an interval  whose length is  as:



**Sampling**

Let us define sampling. Given a signal , we want to estimate it over  intervals using  numbers. Given a number of samples , we define the intervals  as . We will note their equal length as . We want to find a set of numbers  so that the number  estimates the signal over the interval . Let us define the approximation function  where  . We want to find the set of numbers  so that the error would be minimal.

We will solve a more general version of this sampling problem, where the function sampled is of the form . We will say that  and thus  . The intervals will be  , and their length is . The definition of  remains unchanged. By inputting , we get the sampling problem for one dimensional signals we defined above.

First, the optimal sampling in the MSE sense:



Meaning the MSE over the entire interval is the average MSE of the intervals, and we can optimize each interval alone. Let us find the optimal number  for the interval :



We will try to find a minimum of the MSE by derivation:



We want to find the minimum, meaning we want to find  for which ,

meaning , meaning the optimal samples in the MSE sense are the average values of each interval.

Let us show it really is a local minimum using second derivative:



Since the second derivative is positive for it is a local minimum. It is a global minimum since it is the only extremum of the function and the values of  aren't limited.

Now, let us calculate the MSE over an interval using the optimal samples we found:



And the total MSE is:



**Quantization (in the MSE sense only)**

Given a set of optimal samples  for a signal , we still can't represent that on a computer since the numbers  are real and have unlimited precision. Given a representation of a number using  bits, we can only represent  different numbers within the range . These numbers, , are called representation levels. There are several methods to quantize a signal, we will discuss a fairly simple method, assuming the distribution of the signal's values over  is uniform. Since we have b bits, let us divide the interval into  intervals of equals size, whose size is . We now want to find the optimal representation level  for each interval. Assuming the distribution of the values is uniform over , this is exactly the general version of the sampling problem for the function  over the interval . Thus, using the results we got before, the optimal representation level  is:



And the MSE over an interval is:

